Appendix A

RANS Equations for Axisymmetric Swirling Flow

The RANS equations for steady, incompressible flow are presented below in cylindrical-polar co-ordinates, using a stationary reference frame. The velocity components in the radial (r), axial (y) and tangential (ϕ) directions are denoted U, V and W respectively. Convection terms are shown in conservative form. For confirmation of these equations see Owen & Wilson [112] or Morse [124].

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{\partial V}{\partial y} = 0 \tag{A.1}$$

Radial Momentum, U

$$\begin{aligned}\frac{\partial}{\partial r}(\rho rUU) + \frac{\partial}{\partial y}(\rho rUV) - \rho W^2 \quad = \quad & -r\frac{\partial P}{\partial r} + \frac{\partial}{\partial r}\left(2r\mu\frac{\partial U}{\partial r} - \rho r\overline{u^2}\right) \\ & + \frac{\partial}{\partial y}\left[r\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right) - \rho r\overline{uv}\right] \\ & - \left(2\mu\frac{U}{r} - \rho\overline{w^2}\right)\end{aligned} \tag{A.2}$$

Axial Momentum, V

$$\begin{aligned}\frac{\partial}{\partial r}(\rho rUV) + \frac{\partial}{\partial y}(\rho rVV) \quad = \quad & -r\frac{\partial P}{\partial y} + \frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial V}{\partial r} + \frac{\partial U}{\partial y}\right) - \rho r\overline{uv}\right] \\ & + \frac{\partial}{\partial y}\left(2r\mu\frac{\partial V}{\partial y} - \rho r\overline{v^2}\right)\end{aligned} \tag{A.3}$$

Tangential Momentum, W

$$\frac{\partial}{\partial r}(\rho rUW)+\frac{\partial}{\partial y}(\rho rVW)+\rho UW = \frac{\partial}{\partial r}\left(r\mu\frac{\partial W}{\partial r}-\rho r\overline{uw}\right)+\frac{\partial}{\partial y}\left(r\mu\frac{\partial W}{\partial y}-\rho r\overline{vw}\right) - \left(\frac{\mu W}{r}+W\frac{\partial \mu}{\partial r}+\rho\overline{uw}\right) \tag{A.4}$$

A.1 Linear $k-\varepsilon$ Model

Reynolds Stress, $\overline{u_i u_j}$

In a linear $k-\varepsilon$ model the Reynolds stresses, $\overline{u_i u_j}$, are as follows:

$$-\rho\overline{u^2} = 2\mu_t\frac{\partial U}{\partial r}-\frac{2}{3}\rho k \tag{A.5}$$

$$-\rho\overline{v^2} = 2\mu_t\frac{\partial V}{\partial y}-\frac{2}{3}\rho k \tag{A.6}$$

$$-\rho\overline{w^2} = 2\mu_t\frac{U}{r}-\frac{2}{3}\rho k \tag{A.7}$$

$$-\rho\overline{uv} = \mu_t\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right) \tag{A.8}$$

$$-\rho\overline{uw} = \mu_t r\frac{\partial}{\partial r}\left(\frac{W}{r}\right) = \mu_t\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right) \tag{A.9}$$

$$-\rho\overline{vw} = \mu_t\frac{\partial W}{\partial y} \tag{A.10}$$

Substituting these into the above RANS equations:

Radial Momentum, U

$$\frac{\partial}{\partial r}(\rho rUU)+\frac{\partial}{\partial y}(\rho rUV)-\rho W^2 = -r\frac{\partial P}{\partial r}-\frac{\partial}{\partial r}\left(\frac{2}{3}r\rho k\right)+\frac{\partial}{\partial r}\left[2r(\mu+\mu_t)\frac{\partial U}{\partial r}\right] +\frac{\partial}{\partial y}\left[r(\mu+\mu_t)\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right)\right] -2(\mu+\mu_t)\frac{U}{r}+\frac{2}{3}\rho k \tag{A.11}$$

Here the $2k/3$ term resulting from the $\overline{w^2}$ Reynolds stress cancels with part of the expanded $\overline{u^2}$ term:

$$-\frac{\partial}{\partial r}\left(\frac{2}{3}r\rho k\right)+\frac{2}{3}\rho k = -r\frac{\partial}{\partial r}\left(\frac{2}{3}\rho k\right)-\frac{2}{3}\rho k\frac{\partial r}{\partial r}+\frac{2}{3}\rho k = -r\frac{\partial}{\partial r}\left(\frac{2}{3}\rho k\right) \tag{A.12}$$

The remaining gradient of $2k/3$ is included in the pressure gradient term when the transport equations are coded. The final form of the radial-momentum transport equation is then:

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U U) + \frac{\partial}{\partial y}(\rho r U V) &= -r\frac{\partial P'}{\partial r} + \frac{\partial}{\partial r}\left(2r\mu_{eff}\frac{\partial U}{\partial r}\right) \\ &+ \frac{\partial}{\partial y}\left[r\mu_{eff}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right)\right] - 2\mu_{eff}\frac{U}{r} + \rho W^2 \end{aligned} \quad \text{(A.13)}$$

where $\mu_{eff} = \mu + \mu_t$ and $P' = P + 2\rho k/3$.

Axial Momentum, V

Following a similar approach outlined above, the axial momentum expression can be written:

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U V) + \frac{\partial}{\partial y}(\rho r V V) &= -r\frac{\partial P'}{\partial y} + \frac{\partial}{\partial r}\left[r\mu_{eff}\left(\frac{\partial V}{\partial r} + \frac{\partial U}{\partial y}\right)\right] \\ &+ \frac{\partial}{\partial y}\left(2r\mu_{eff}\frac{\partial V}{\partial y}\right) \end{aligned} \quad \text{(A.14)}$$

Tangential Momentum, W

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U W) + \frac{\partial}{\partial y}(\rho r V W) + \rho U W &= \frac{\partial}{\partial r}\left(r\mu\frac{\partial W}{\partial r} + r\mu_t\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right) + \frac{\partial}{\partial y}\left(r\mu_{eff}\frac{\partial W}{\partial y}\right) \\ &- \left(\mu\frac{W}{r} + W\frac{\partial \mu}{\partial r} - \mu_t\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right) \end{aligned} \quad \text{(A.15)}$$

This can be rearranged as follows:

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U W) + \frac{\partial}{\partial y}(\rho r V W) &= \frac{\partial}{\partial r}\left(r\mu_{eff}\frac{\partial W}{\partial r}\right) + \frac{\partial}{\partial y}\left(r\mu_{eff}\frac{\partial W}{\partial y}\right) \\ &- \mu_{eff}\frac{W}{r} - W\frac{\partial \mu_{eff}}{\partial r} - \rho U W \end{aligned} \quad \text{(A.16)}$$

Kinetic energy, k

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U k) + \frac{\partial}{\partial y}(\rho r V k) &= \frac{\partial}{\partial r}\left[r\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial r}\right] + \frac{\partial}{\partial y}\left[r\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] \\ &+ rP_k - r\rho\varepsilon \end{aligned} \quad \text{(A.17)}$$

where the production rate, P_k, is given by:

$$\begin{aligned}
P_k &= \mu_t S_{ij}\frac{\partial U_i}{\partial x_j} \\
&= \mu_t\left\{2\left(\frac{\partial U}{\partial r}\right)^2+2\left(\frac{\partial V}{\partial y}\right)^2+2\left(\frac{U}{r}\right)^2\right. \\
&\quad \left.+\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right\} \qquad \text{(A.18)}
\end{aligned}$$

and the total dissipation rate, ε, comprising of isotropic dissipation rate ($\tilde{\varepsilon}$) and the value of dissipation rate at the wall, is given by:

$$\begin{aligned}
\varepsilon &= \tilde{\varepsilon}+2\nu\left(\frac{\partial\sqrt{k}}{\partial x_j}\right)^2 \\
&= \tilde{\varepsilon}+2\nu\left[\left(\frac{\partial\sqrt{k}}{\partial r}\right)^2+\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2\right] \qquad \text{(A.19)}
\end{aligned}$$

Dissipation Rate, $\tilde{\varepsilon}$

$$\begin{aligned}
\frac{\partial}{\partial r}(r\rho U\tilde{\varepsilon})+\frac{\partial}{\partial y}(\rho rV\tilde{\varepsilon}) &= \frac{\partial}{\partial r}\left[r\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\tilde{\varepsilon}}{\partial r}\right]+\frac{\partial}{\partial y}\left[r\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\tilde{\varepsilon}}{\partial y}\right] \\
&\quad +r\left(c_{\varepsilon1}f_1P_k-c_{\varepsilon2}f_2\rho\tilde{\varepsilon}\right)\frac{\tilde{\varepsilon}}{k}+rP_{\varepsilon3}+r\rho Y_c \qquad \text{(A.20)}
\end{aligned}$$

where the gradient production term, $P_{\varepsilon3}$, is given by:

$$\begin{aligned}
P_{\varepsilon3} &= 2\mu\nu_t\left(\frac{\partial^2 U_i}{\partial x_j\partial x_k}\right)^2 \\
&= 2\mu\nu_t\left\{\left(\frac{\partial^2 U}{\partial r^2}\right)^2+3\left[\frac{1}{r^2}\left(r\frac{\partial U}{\partial r}-U\right)\right]^2+\left(\frac{\partial^2 U}{\partial y^2}\right)^2\right. \\
&\quad +2\left(\frac{\partial^2 U}{\partial y\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial W}{\partial y}\right)^2 \\
&\quad +\left(\frac{\partial^2 W}{\partial r^2}\right)^2+3\left[\frac{1}{r^2}\left(r\frac{\partial W}{\partial r}-W\right)\right]^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2 \\
&\quad +2\left(\frac{\partial^2 W}{\partial y\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial U}{\partial y}\right)^2 \\
&\quad +\left(\frac{\partial^2 V}{\partial r^2}\right)^2+\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)^2+\left(\frac{\partial^2 V}{\partial y^2}\right)^2 \\
&\quad \left.+2\left(\frac{\partial^2 V}{\partial y\partial r}\right)^2\right\} \qquad \text{(A.21)}
\end{aligned}$$

The whole of the above expression has been used for the work included in this thesis. Morse [124] and Launder & Sharma [13] used a simplified form of the above $P_{\varepsilon 3}$ expression.

A.2 Non-Linear $k-\varepsilon$ Model

In axisymmetric swirling flows the strain-rate and vorticity tensors appearing in the NLEVM are given by:

$$\begin{aligned} S_{ij} &= \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \\ &= \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \\ &= \begin{bmatrix} \left(2\frac{\partial U}{\partial r}\right) & \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right) & \left(\frac{\partial W}{\partial r} - \frac{W}{r}\right) \\ \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right) & \left(2\frac{\partial V}{\partial y}\right) & \left(\frac{\partial W}{\partial y}\right) \\ \left(\frac{\partial W}{\partial r} - \frac{W}{r}\right) & \left(\frac{\partial W}{\partial y}\right) & \left(2\frac{U}{r}\right) \end{bmatrix} \end{aligned} \tag{A.22}$$

and:

$$\begin{aligned} \Omega_{ij} &= \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \\ &= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial r}\right) & -\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right) \\ -\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial r}\right) & 0 & -\left(\frac{\partial W}{\partial y}\right) \\ \left(\frac{\partial W}{\partial r} + \frac{W}{r}\right) & \left(\frac{\partial W}{\partial y}\right) & 0 \end{bmatrix} \end{aligned} \tag{A.23}$$