Appendix A

RANS Equations for Axisymmetric Swirling Flow

The RANS equations for steady, incompressible flow are presented below in cylindrical-polar coordinates, using a stationary reference frame. The velocity components in the radial ($r$), axial ($y$) and tangential ($\phi$) directions are denoted $U$, $V$ and $W$ respectively. Convection terms are shown in conservative form. For confirmation of these equations see Owen & Wilson [112] or Morse [124].

Continuity

$$\frac{1}{r} \frac{\partial}{\partial r} (rU) + \frac{\partial V}{\partial y} = 0 \quad (A.1)$$

Radial Momentum, $U$

$$\frac{\partial}{\partial r} (prU) + \frac{\partial}{\partial y} (prUV) - \rho W^2 = -r \frac{\partial P}{\partial r} + \frac{\partial}{\partial r} \left( 2\mu \frac{\partial U}{\partial r} - \rho u^2 \right) + \frac{\partial}{\partial y} \left[ \rho \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) - \rho r \mu \frac{\partial U}{\partial y} \right] - \left( 2\mu \frac{U}{r} - \rho \frac{W^2}{r} \right) \quad (A.2)$$

Axial Momentum, $V$

$$\frac{\partial}{\partial r} (prUV) + \frac{\partial}{\partial y} (prVV) = -r \frac{\partial P}{\partial y} + \frac{\partial}{\partial r} \left[ \rho \mu \left( \frac{\partial V}{\partial r} + \frac{\partial U}{\partial y} \right) - \rho r \mu \frac{V}{r} \right] + \frac{\partial}{\partial y} \left( 2\mu \frac{\partial V}{\partial y} - \rho \frac{V^2}{r} \right) \quad (A.3)$$
A.1 Linear $k - \varepsilon$ Model

Tangential Momentum, $W$

\[
\frac{\partial}{\partial r} (\rho r U W) + \frac{\partial}{\partial y} (\rho r V W) + \rho W = \frac{\partial}{\partial r} \left( r \mu \frac{\partial W}{\partial r} - \rho r \overline{uw} \right) + \frac{\partial}{\partial y} \left( r \mu \frac{\partial W}{\partial y} - \rho r \overline{vw} \right)
\]

\[
- \left( \frac{\mu W}{r} + W \frac{\partial}{\partial r} + \rho \overline{ww} \right)
\]

\[ (A.4) \]

A.1 Linear $k - \varepsilon$ Model

Reynolds Stress, $\overline{u_i u_j}$

In a linear $k - \varepsilon$ model the Reynolds stresses, $\overline{u_i u_j}$, are as follows:

\[-\overline{u_i u_i} = 2 \mu \frac{\partial U}{\partial r} - \frac{2}{3} \rho k \]

\[ (A.5) \]

\[-\overline{v_i v_i} = 2 \mu \frac{\partial V}{\partial y} - \frac{2}{3} \rho k \]

\[ (A.6) \]

\[-\overline{w_i w_i} = 2 \mu \frac{U}{r} - \frac{2}{3} \rho k \]

\[ (A.7) \]

\[-\overline{u_i v_j} = \mu \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) \]

\[ (A.8) \]

\[-\overline{u_i w_j} = \mu \frac{\partial}{\partial r} \left( \frac{W}{r} \right) = \mu \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \]

\[ (A.9) \]

\[-\overline{v_i w_j} = \mu \frac{\partial W}{\partial y} \]

\[ (A.10) \]

Substituting these into the above RANS equations:

Radial Momentum, $U$

\[
\frac{\partial}{\partial r} (\rho r U U) + \frac{\partial}{\partial y} (\rho r U V) - \rho W^2 = -r \frac{\partial P}{\partial r} - \frac{\partial}{\partial r} \left( \frac{2}{3} \rho k \right) + \frac{\partial}{\partial r} \left[ 2r(\mu + \mu_t) \frac{\partial U}{\partial r} \right]
\]

\[
+ \frac{\partial}{\partial y} \left[ r(\mu + \mu_t) \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) \right]
\]

\[-2(\mu + \mu_t) \frac{U}{r} + \frac{2}{3} \rho k \]

\[ (A.11) \]

Here the $2k/3$ term resulting from the $\overline{w^2}$ Reynolds stress cancels with part of the expanded $\overline{u^2}$ term:

\[-r \frac{\partial}{\partial r} \left( \frac{2}{3} \rho k \right) + \frac{2}{3} \rho k = -r \frac{\partial}{\partial r} \left( \frac{2}{3} \rho k \right) - \frac{2}{3} \rho k \frac{\partial r}{\partial r} + \frac{2}{3} \rho k \]

\[ = -r \frac{\partial}{\partial r} \left( \frac{2}{3} \rho k \right) \]

\[ (A.12) \]
APPENDIX A. RANS Equations for Axisymmetric Swirling Flow

The remaining gradient of $2k/3$ is included in the pressure gradient term when the transport equations are coded. The final form of the radial-momentum transport equation is then:

$$
\frac{\partial}{\partial r} (\rho r U U) + \frac{\partial}{\partial y} (\rho r U V) = -r \frac{\partial P'}{\partial r} + \frac{\partial}{\partial r} \left( 2\mu_{eff} \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial y} \left( \mu_{eff} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) \right) - 2\mu_{eff} \frac{U}{r} + \rho W^2
$$

(A.13)

where $\mu_{eff} = \mu + \mu_t$ and $P' = P + 2pk/3$.

Axial Momentum, $V$

Following a similar approach outlined above, the axial momentum expression can be written:

$$
\frac{\partial}{\partial r} (\rho r U V) + \frac{\partial}{\partial y} (\rho r V V) = -r \frac{\partial P'}{\partial y} + \frac{\partial}{\partial r} \left( \mu_{eff} \left( \frac{\partial V}{\partial r} + \frac{\partial U}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left( 2\mu_{eff} \frac{\partial V}{\partial y} \right)
$$

(A.14)

Tangential Momentum, $W$

$$
\frac{\partial}{\partial r} (\rho r U W) + \frac{\partial}{\partial y} (\rho r V W) + \rho W = \frac{\partial}{\partial r} \left( \mu \frac{\partial W}{\partial r} + r \mu_t \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \right) + \frac{\partial}{\partial y} \left( \mu_{eff} \frac{\partial W}{\partial y} \right)
$$

$$
- \left( \frac{W}{r} - W \frac{\partial \mu}{\partial r} - \mu_t \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \right)
$$

(A.15)

This can be rearranged as follows:

$$
\frac{\partial}{\partial r} (\rho r U W) + \frac{\partial}{\partial y} (\rho r V W) = \frac{\partial}{\partial r} \left( \mu_{eff} \frac{\partial W}{\partial r} \right) + \frac{\partial}{\partial y} \left( \mu_{eff} \frac{\partial W}{\partial y} \right)
$$

$$
- \mu_{eff} \frac{W}{r} - W \frac{\partial \mu_{eff}}{\partial r} - \rho U W
$$

(A.16)

Kinetic energy, $k$

$$
\frac{\partial}{\partial r} (\rho r U k) + \frac{\partial}{\partial y} (\rho r V k) = \frac{\partial}{\partial r} \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial r} + \frac{\partial}{\partial y} \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial y}
$$

$$
+ rP_k - r \rho \epsilon
$$

(A.17)
where the production rate, $P_k$, is given by:

$$P_k = \mu_S i_j \frac{\partial U}{\partial x_j}$$

$$= \mu_t \left\{ 2 \left( \frac{\partial U}{\partial r} \right)^2 + 2 \left( \frac{\partial V}{\partial y} \right)^2 + 2 \left( \frac{U}{r} \right)^2 + \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right)^2 + \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right\} \quad (A.18)$$

and the total dissipation rate, $\varepsilon$, comprising of isotropic dissipation rate ($\bar{\varepsilon}$) and the value of dissipation rate at the wall, is given by:

$$\varepsilon = \bar{\varepsilon} + 2\nu \left[ \left( \frac{\partial \sqrt{k}}{\partial x_j} \right)^2 + \left( \frac{\partial \sqrt{k}}{\partial y} \right)^2 \right] \quad (A.19)$$

**Dissipation Rate, $\bar{\varepsilon}$**

$$\frac{\partial}{\partial r}(rp\bar{\varepsilon}) + \frac{\partial}{\partial y}(r\rho \bar{\varepsilon}) = \frac{\partial}{\partial r} \left[ r \left( \mu + \frac{\mu_t}{\sigma_t} \right) \frac{\partial \bar{\varepsilon}}{\partial r} \right] + \frac{\partial}{\partial y} \left[ r \left( \mu + \frac{\mu_t}{\sigma_t} \right) \frac{\partial \bar{\varepsilon}}{\partial y} \right] + r(c_{t1} f_1 p_k - c_{t2} f_2 \bar{p} \bar{\varepsilon}) + rP_{e3} + r\rho Y_c \quad (A.20)$$

where the gradient production term, $P_{e3}$, is given by:

$$P_{e3} = 2\nu \left\{ \left( \frac{\partial^2 U}{\partial x_j \partial x_i} \right)^2 + \left( \frac{1}{r^2} \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \right)^2 + \left( \frac{\partial \bar{\varepsilon}}{\partial y^2} \right)^2 \right\} \quad (A.21)$$
The whole of the above expression has been used for the work included in this thesis. Morse [124] and Launder & Sharma [13] used a simplified form of the above $P_{e3}$ expression.

### A.2 Non-Linear $k - \varepsilon$ Model

In axisymmetric swirling flows the strain-rate and vorticity tensors appearing in the NLEVM are given by:

$$ S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} \left( \frac{2}{r} \frac{\partial U}{\partial r} \right) & \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) & \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \\ \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) & \left( 2 \frac{\partial V}{\partial y} \right) & \left( \frac{\partial W}{\partial y} \right) \\ \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) & \left( \frac{\partial W}{\partial y} \right) & \left( \frac{2}{r} \frac{U}{r} \right) \end{bmatrix} \quad (A.22) $$

and:

$$ \Omega_{ij} = \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \left( \frac{\partial U}{\partial y} - \frac{\partial V}{\partial r} \right) & \left( \frac{\partial W}{\partial r} + \frac{W}{r} \right) \\ -\left( \frac{\partial U}{\partial y} - \frac{\partial V}{\partial r} \right) & 0 & \left( \frac{\partial W}{\partial y} \right) \\ \left( \frac{\partial W}{\partial r} + \frac{W}{r} \right) & \left( \frac{\partial W}{\partial y} \right) & 0 \end{bmatrix} \quad (A.23) $$