Appendix C

RANS Equations in Curvilinear Coordinates

To begin with, the Reynolds-averaged Navier-Stokes (RANS) equations are presented in the familiar vector and Cartesian tensor forms. Each term in the transport equations is examined to see whether it complies with the Einstein summation convention for curvilinear coordinates. Then, according to the rules outlined in Appendix B, the equations are re-written in non-orthogonal curvilinear coordinates in which velocity vectors follow the coordinate directions. Those equations which are used in the derivation of the UMIST-N wall function transport equations are shown in boxes to highlight their significance.

C.1 Vector Form

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \tag{C.1}$$

Momentum

$$\frac{\partial}{\partial t}(\rho \mathbf{U}) + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U} - \mathbf{T}) = -\nabla P \tag{C.2}$$

Scalar, ϕ

$$\frac{\partial}{\partial t}(\rho \phi) + \nabla \cdot (\rho \mathbf{U} \phi - \mathbf{q}) \quad = \quad S_\phi \tag{C.3}$$

where $\mathbf{U}$ and $\mathbf{q}$ are, respectively, the mean velocity and scalar-flux vectors, $\mathbf{T}$ is the second-order stress tensor $\left(\mathbf{T} = \tau^{ij} \mathbf{g}_i \otimes \mathbf{g}_j\right)$ and P is the mean pressure. The divergence of a vector quantity is a scalar whilst the divergence of a second-order tensor results in a vector quantity and, therefore, in 3-D space the momentum equation consists of three component equations each multiplied by a base vector.

C.2 Cartesian Coordinates

The continuity, momentum and scalar transport equations can be written in Cartesian tensors as follows:

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}\left(\rho U^j\right) = 0 \tag{C.4}$$

Momentum

$$\frac{\partial}{\partial t}\left(\rho U^i\right) + \frac{\partial}{\partial x^j}\left(\rho U^i U^j - \tau^{ij}\right) = -\frac{\partial P}{\partial x^i} \tag{C.5}$$

Scalar, ϕ

$$\frac{\partial}{\partial t}\left(\rho \phi\right) + \frac{\partial}{\partial x^j}\left(\rho \phi U^j - J^j_\phi\right) = S^i_\phi \tag{C.6}$$

Turbulent Kinetic Energy, k

$$\frac{\partial}{\partial t}\left(\rho k\right) + \frac{\partial}{\partial x^j}\left(\rho k U^j - \frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x^j}\right) = G - \rho\varepsilon \tag{C.7}$$

Isotropic Dissipation Rate, $\tilde{\varepsilon}$

$$\frac{\partial}{\partial t}\left(\rho \tilde{\varepsilon}\right) + \frac{\partial}{\partial x^j}\left(\rho \tilde{\varepsilon} U^j - \frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \tilde{\varepsilon}}{\partial x^j}\right) = c_{\varepsilon 1} f_1 G\frac{\tilde{\varepsilon}}{k} - c_{\varepsilon 2} f_2 \rho\frac{\tilde{\varepsilon}^2}{k} + \rho Y_c + P_{\varepsilon 3} \tag{C.8}$$

where the stress tensor, τ^{ij}, is given by:

$$\tau^{ij} = \mu\left(\frac{\partial U^i}{\partial x^j} + \frac{\partial U^j}{\partial x^i} - \underbrace{\frac{2}{3}\delta^{ij}\frac{\partial U^m}{\partial x^m}}\right) - \rho\overline{u^i u^j} \tag{C.9}$$

In an incompressible flow, the underbraced terms in the above equations disappears since, from continuity, $(\partial U^m/\partial x^m = 0)$. The Reynolds stress $\left(-\rho\overline{u^i u^j}\right)$, is expressed, using a linear eddy-viscosity model:

$$-\rho\overline{u^i u^j} = \mu_t\left(\frac{\partial U^i}{\partial x^j} + \frac{\partial U^j}{\partial x^i} - \underbrace{\frac{2}{3}\delta^{ij}\frac{\partial U^m}{\partial x^m}}\right) - \underbrace{\frac{2}{3}\delta^{ij}\rho k} \tag{C.10}$$

where the two underbraced terms in the above equation arise from the trace condition (where, by definition, $\overline{u^i u^i} = 2k$). The term, J^j_ϕ, representing molecular and turbulent diffusive flux of the time-averaged scalar, ϕ, is given by:

$$J_\phi^j = \Gamma_\phi \frac{\partial \phi}{\partial x^j} - \rho\overline{u^j \phi} \tag{C.11}$$

where Γ_ϕ is the molecular diffusivity (not to be confused with the Christoffel symbol, Γ^i_{jk}, which has three indices). The turbulent scalar flux $\left(-\rho\overline{u^j \phi}\right)$ is calculated using an eddy-diffusivity model:

$$-\rho\overline{u^j \phi} = \frac{\mu_t}{\sigma_\phi}\frac{\partial \phi}{\partial x^j} \tag{C.12}$$

The eddy-viscosity, μ_t, using a $k - \varepsilon$ turbulence model, is given by:

$$\mu_t = \rho c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}} \tag{C.13}$$

the production rate of turbulent kinetic energy, G:

$$G = -\rho\overline{u^i u^j}\frac{\partial U^i}{\partial x^j} \tag{C.14}$$

the total dissipation rate, ε:

$$\varepsilon = \tilde{\varepsilon} + 2\nu \left(\frac{\partial k^{1/2}}{\partial x^j}\right)^2 \tag{C.15}$$

the Yap correction, Y_c:

$$Y_c = \max\left\{\left[0.83\left(\frac{k^{3/2}/\tilde{\varepsilon}}{2.55y} - 1\right)\left(\frac{k^{3/2}/\tilde{\varepsilon}}{2.55y}\right)^2 \frac{\tilde{\varepsilon}^2}{k}\right], 0\right\} \tag{C.16}$$

and the gradient production term, $P_{\varepsilon 3}$:

$$P_{\varepsilon 3} = 2\mu\nu_t \left(\frac{\partial^2 U^i}{\partial x^j \partial x^k}\right)^2 \tag{C.17}$$

C.3 Summation Convention

The derivatives appearing in the above transport equations in Cartesian coordinates are *covariant*, in which case the derivative of, say, ϕ with respect to x^j can be written, $\partial\phi/\partial x^j = \phi_{,j}$, which in itself is a covariant tensor. Recalling Section B.19, the summation convention in general coordinates is defined such that summation only occurs where one dummy index is subscript and the other is superscript (i.e. summation only applies between covariant and contravariant terms). The second covariant derivative of ϕ cannot be written $\left(\phi_{,j}\right)_{,j}$ since this would violate the summation convention as both dummy indices are subscript. In order to change $\phi_{,j}$ into a contravariant tensor (effectively raising index j from subscript to superscript), the product is taken with the contravariant metric tensor g^{ij} (which in Cartesian coordinates is equivalent to the Kronecker delta, δ^{ij}). As was shown in Section B.19, the correct form of the second derivative of ϕ is thus $\left(\delta^{jm}\phi_{,m}\right)_{,j}$ where dummy indices j and m are now repeated as subscript and superscript. Additionally, all of the terms in the each equation should be consistently covariant or contravariant, for instance, one cannot have a contravariant component on one side of the equation equal to a covariant component on the other. In the following sections, each of the terms in the momentum, kinetic energy and dissipation rate equations are examined to see whether they comply with the summation convention.

Convection

In the conservative form of convection of momentum, given by:

$$\frac{\partial\left(\rho U^i U^j\right)}{\partial x^j} = \left(\rho U^i U^j\right)_{,j} \tag{C.18}$$

the suffix j appears in the contravariant velocity component and the covariant derivative and therefore complies with the convention. Likewise, in the non-conservative form of convection:

$$\rho U^j \frac{\partial U^i}{\partial x^j} = \rho U^j \left(U^i\right)_{,j} \tag{C.19}$$

index j is repeated in subscript and superscript. Convection of a scalar quantity (ϕ, k or $\tilde{\varepsilon}$) can also be shown to agree with the summation convection.

Diffusion

The covariant gradient of the contravariant stress, τ^{ij}, is written:

$$\frac{\partial \tau^{ij}}{\partial x^j} = \left(\tau^{ij}\right)_{,j} \tag{C.20}$$

where j is repeated in subscript and superscript, in agreement with the summation convention.

Stress Tensor, τ^{ij}

In the contravariant stress tensor:

$$\tau^{ij} = \mu\left(\frac{\partial U^i}{\partial x^j} + \frac{\partial U^j}{\partial x^i} - \frac{2}{3}\delta^{ij}\frac{\partial U^m}{\partial x^m}\right) - \rho\overline{u^i u^j} \tag{C.21}$$

the strain-rate $\left(\partial U^i/\partial x^j\right) = U^i_{,j}$ involves a superscript i and subscript j. Since both i and j must be contravariant (since τ^{ij} is fully contravariant), the Kronecker delta, δ^{jm}, is introduced as follows:

$$\frac{\partial U^i}{\partial x^j} = U^i_{,j} = \delta^{jm}U^i_{,m} \tag{C.22}$$

where i and j now appear as superscripts. Likewise the second strain-rate term in Equation (C.21) is written:

$$\frac{\partial U^j}{\partial x^i} = U^j_{,i} = \delta^{im} U^j_{,m} \tag{C.23}$$

The remaining terms in Equation (C.21) agree with the summation convention. The contravariant stress tensor, τ^{ij}, is therefore given by:

$$\tau^{ij} = \mu \left(\delta^{jm} \frac{\partial U^i}{\partial x^m} + \delta^{im} \frac{\partial U^j}{\partial x^m} - \frac{2}{3} \delta^{ij} \frac{\partial U^m}{\partial x^m} \right) - \rho \overline{u^i u^j} \tag{C.24}$$

Scalar Flux Vector

Following a similar reasoning to that used with the stress tensor, the scalar flux J^j_ϕ, which is modelled using the eddy-diffusivity hypothesis, is given by:

$$\begin{aligned} J^j_\phi &= \Gamma_\phi \frac{\partial \phi}{\partial x^j} - \rho \overline{u^j \phi} \\ &= \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi} \right) \frac{\partial \phi}{\partial x^j} \\ &= \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi} \right) \phi_{,j} \end{aligned} \tag{C.25}$$

Since the scalar flux should be a contravariant vector, the subscript j must be raised by multiplying the above expression with δ^{jm}:

$$J^j_\phi = \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi} \right) \delta^{jm} \frac{\partial \phi}{\partial x^m} = \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi} \right) \delta^{jm} \phi_{,m} \tag{C.26}$$

Pressure Gradient

The pressure gradient in Equation (C.5) is expressed as the covariant derivative of the scalar parameter P and is therefore itself a covariant tensor, $\left(\partial P / \partial x^i = P_{,i}\right)$. However, the convection and diffusion terms, discussed above, both yield contravariant components once summation has been applied over the dummy indices (i.e. both convection and diffusion terms have superscript i). Therefore, to obtain agreement with all the components in the momentum equation, it is necessary to to raise the index i in the pressure gradient:

$$\frac{\partial P}{\partial x^i} = P_{,i} = \delta^{ij} P_{,j} \tag{C.27}$$

Turbulent Kinetic Energy Production, G

The production source term G in the k- and $\tilde{\varepsilon}$-equations is given by:

$$G = -\rho \overline{u^i u^j} \frac{\partial U^i}{\partial x^j} = -\rho \overline{u^i u^j} U^i_{,j} \tag{C.28}$$

Here the dummy index j is repeated in the subscript and superscript, in agreement with the summation convention, but index i appears superscript in both instances. Therefore, to satisfy the summation convention for index i, the covariant Kronecker delta (δ_{im}) is introduced, as follows:

$$G = -\rho\delta_{im}\overline{u^m u^j}U^i_{,j} \tag{C.29}$$

where the dummy indices (i, j and m) are all repeated in subscript and superscript.

Total Dissipation Rate, ε

The total dissipation rate, ε, is given by:

$$\varepsilon = \tilde{\varepsilon} + 2\nu\left(\frac{\partial k^{1/2}}{\partial x^j}\right)^2 = \tilde{\varepsilon} + 2\nu\left(k^{1/2}\right)_{,j}\left(k^{1/2}\right)_{,j} \tag{C.30}$$

This does not agree with the summation convention in general coordinates since both derivatives of $k^{1/2}$ are covariant tensors. To make one of the tensors contravariant, the contravariant Kronecker delta $\left(\delta^{jm}\right)$ is introduced, as follows:

$$\varepsilon = \tilde{\varepsilon} + 2\nu\delta^{jm}\left(k^{1/2}\right)_{,m}\left(k^{1/2}\right)_{,j} \tag{C.31}$$

Gradient Production of Dissipation Rate, $P_{\varepsilon 3}$

In the low-Reynolds-number model $\tilde{\varepsilon}$-equation, the near-wall damping term ($P_{\varepsilon 3}$) is given by:

$$P_{\varepsilon 3} = 2\mu\nu_t\left(\frac{\partial^2 U^i}{\partial x^j \partial x^k}\right)^2 = 2\mu\nu_t\left(U^i_{,j}\right)_{,k}\left(U^i_{,j}\right)_{,k} \tag{C.32}$$

In order to obtain the second covariant derivative one must first raise the index of the first derivative to satisfy the summation convention. The second derivative of the velocity is therefore written:

$$\left(\delta^{jk}U^i_{,k}\right)_{,l} \tag{C.33}$$

where k is a dummy index. The square of this term requires the introduction of three Kronecker deltas to comply with the need for repeated raised and lowered indices:

$$P_{\varepsilon 3} = 2\mu\nu_t\delta_{im}\delta_{jn}\delta^{lp}\left(\delta^{no}U^m_{,o}\right)_{,p}\left(\delta^{jk}U^i_{,k}\right)_{,l} \tag{C.34}$$

Unfortunately, to agree with the summation convention, $P_{\varepsilon 3}$ now contains eight dummy indices (i, j, k, l, m, n, o and p).

To summarize the preceding sections, the momentum and scalar transport equations are re-written below in Cartesian coordinates, where the general coordinate summation convention is observed:

Momentum

$$\frac{\partial}{\partial t}\left(\rho U^i\right)+\frac{\partial}{\partial x^j}\left(\rho U^i U^j-\tau^{ij}\right)=-\delta^{ij}\frac{\partial P}{\partial x^j} \tag{C.35}$$

where the stress tensor is given by:

$$\tau^{ij}=\mu\left(\delta^{jm}\frac{\partial U^i}{\partial x^m}+\delta^{im}\frac{\partial U^j}{\partial x^m}-\frac{2}{3}\delta^{ij}\frac{\partial U^m}{\partial x^m}\right)-\rho\overline{u^i u^j} \tag{C.36}$$

and the Reynolds stress:

$$-\rho\overline{u^i u^j}=\mu_t\left(\delta^{jm}\frac{\partial U^i}{\partial x^m}+\delta^{im}\frac{\partial U^j}{\partial x^m}-\frac{2}{3}\delta^{ij}\frac{\partial U^m}{\partial x^m}\right)-\frac{2}{3}\delta^{ij}\rho k \tag{C.37}$$

Scalar

$$\frac{\partial}{\partial t}\left(\rho\phi\right)+\frac{\partial}{\partial x^j}\left[\rho\phi U^j-\left(\Gamma_\phi+\frac{\mu_t}{\sigma_\phi}\right)\delta^{jm}\frac{\partial\phi}{\partial x^m}\right]=S^i_\phi \tag{C.38}$$

Alternative Approach to Summation Convention

An alternative way of examining the summation convention in the above equations is to start from the vector form of transport equations and convert each term into non-orthogonal curvilinear coordinates directly (omitting the in-between step of Cartesian tensors). For instance, the velocity gradient component of the stress tensor can be written in vector form:

$$\mathbf{T}=\operatorname{grad}\mathbf{v}=\nabla\mathbf{v} \tag{C.39}$$

This can be converted into generalized tensor form as follows (see also Section B.18)

$$\mathbf{T} \;=\; \mathbf{g}^j\frac{\partial\left(v^i\mathbf{g}_i\right)}{\partial\xi^j} \tag{C.40}$$

The covariant derivative of vector $\mathbf{v}$ was shown in Section B.14 to be given by:

$$\frac{\partial\left(v^i\mathbf{g}_i\right)}{\partial\xi^j}=\left(\frac{\partial v^i}{\partial\xi^j}+v^m\Gamma^i_{mj}\right)\mathbf{g}_i=v^i_{,j}\mathbf{g}_i \tag{C.41}$$

hence the expression for the stress tensor becomes:

$$\mathbf{T}=v^i_{,j}\mathbf{g}_i\otimes\mathbf{g}^j \tag{C.42}$$

In the above equation, the stress tensor is expressed in terms of mixed covariant and contravariant base vectors, $\mathbf{g}_i$ and $\mathbf{g}^j$. In order to obtain agreement with the remaining terms in the momentum equation one needs to express the stress tensor in terms of the covariant base vectors $\mathbf{g}_i$ and $\mathbf{g}_j$. To do so, the contravariant metric g^{jk} is introduced to lower the base vector $\mathbf{g}^j$ to $\mathbf{g}_k$:

$$\mathbf{T} = g^{jk} v^i_{,j} \mathbf{g}_i \otimes \mathbf{g}_k \tag{C.43}$$

Switching around the indices j and k so that the stress tensor is in terms of the base vectors $\mathbf{g}_i$ and $\mathbf{g}_j$, one obtains:

$$\mathbf{T} = g^{jk} v^i_{,k} \mathbf{g}_i \otimes \mathbf{g}_j \tag{C.44}$$

The components of the above expression are identical to the first term in the strain-rate tensor obtained in Equation (C.24).

C.4 Transformation Rules

To convert the RANS equations from Cartesian to curvilinear non-orthogonal coordinates, the following set of transformations are applied. These rules are identical to those adopted by Demirdžić *et al.* [86] with the exception that here the transformations are to non-physical parameters[1]. Expressions in Cartesian coordinates are shown on the left and equivalent expressions in non-orthogonal curvilinear coordinates on the right. The use of ∇_j and Δ/Δ_j symbols is to allow direct comparison with the equations given by Demirdžić *et al.* [86].

Scalars

$$\phi \to \phi \tag{C.45}$$

$$\frac{\partial \phi}{\partial x^j} \to \frac{\partial \phi}{\partial \xi^j} \tag{C.46}$$

Vectors

$$v^i \to v^i \tag{C.47}$$

$$\frac{\partial v^i}{\partial x^j} \to \nabla_j v^i = v^i_{,j} = \frac{\partial v^i}{\partial \xi^j} + v^m \Gamma^i_{mj} \tag{C.48}$$

$$\frac{\partial v^j}{\partial x^j} \to \nabla_j v^j = v^j_{,j} \quad = \quad \frac{\Delta v^j}{\Delta \xi^j} \tag{C.49}$$

$$= \quad \frac{1}{J} \frac{\partial}{\partial \xi^j} \left(J v^j \right) \tag{C.50}$$

[1]Demirdžić *et al.* [86] expressed the transformed equations in physical curvilinear coordinates and subsequently integrated the transport equations in physical space (i.e. over the cell dimensions in terms of x^i rather than ξ^i) - for details see [85]. In the treatment outlined in this document, the integration takes place in ξ^i-space and therefore the equations are expressed in terms of the non-physical ξ^i components.

Tensors

$$T^{ij} \rightarrow T^{i} \tag{C.51}$$

$$\delta^{ij} \rightarrow g^{ij} \qquad \delta_{ij} \rightarrow g_{ij} \tag{C.52}$$

$$\begin{aligned}
\frac{\partial T^{ij}}{\partial x^{j}} \rightarrow \nabla_j T^{ij} = T^{ij}_{,j} &= \frac{\Delta T^{ij}}{\Delta \xi^{j}} + T^{mj}\Gamma^{i}_{mj} && \text{(C.53)}\\
&= \frac{\partial T^{ij}}{\partial \xi^{j}} + T^{im}\Gamma^{j}_{mj} + T^{mj}\Gamma^{i}_{mj} && \text{(C.54)}\\
&= \frac{1}{J}\frac{\partial}{\partial \xi^{j}}\left(JT^{ij}\right) + T^{mj}\Gamma^{i}_{mj} && \text{(C.55)}
\end{aligned}$$

C.5 Non-Orthogonal Curvilinear Coordinates

Following the transformation rules summarized in Section C.4, the Navier-Stokes equations described in the preceding section are given below in non-physical non-orthogonal curvilinear coordinates.

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\Delta}{\Delta \xi^{j}}\left(\rho U^{j}\right) = 0 \tag{C.56}$$

Momentum

$$\frac{\partial}{\partial t}\left(\rho U^{i}\right) + \nabla_j\left(\rho U^{i}U^{j} - \tau^{ij}\right) = -g^{ij}\frac{\partial P}{\partial \xi^{j}} \tag{C.57}$$

where the non-physical stress tensor, τ^{ij}, is given by:

$$\tau^{ij} = \mu\left(g^{jm}\nabla_m U^{i} + g^{im}\nabla_m U^{j} - \frac{2}{3}g^{ij}\frac{\Delta U^{m}}{\Delta \xi^{m}}\right) - \rho\overline{u^{i}u^{j}} \tag{C.58}$$

and the non-physical Reynolds stress $\left(-\rho\overline{u^{i}u^{j}}\right)$:

$$-\rho\overline{u^{i}u^{j}} = \mu_t\left(g^{jm}\nabla_m U^{i} + g^{im}\nabla_m U^{j} - \frac{2}{3}g^{ij}\frac{\Delta U^{m}}{\Delta \xi^{m}}\right) - \frac{2}{3}g^{ij}\rho k \tag{C.59}$$

Scalar

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\Delta}{\Delta \xi^{j}}\left[\rho\phi U^{j} - \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi}\right)g^{jm}\frac{\partial \phi}{\partial \xi^{m}}\right] = S_\phi \tag{C.60}$$

Turbulent Kinetic Energy

$$\frac{\partial}{\partial t}(\rho k) + \frac{\Delta}{\Delta \xi^j}\left[\rho k U^j - \left(\mu + \frac{\mu_t}{\sigma_k}\right) g^{jm} \frac{\partial k}{\partial \xi^m}\right] = G - \rho\varepsilon \tag{C.61}$$

Isotropic Dissipation Rate

$$\frac{\partial}{\partial t}(\rho \tilde{\varepsilon}) + \frac{\Delta}{\Delta \xi^j}\left[\rho \tilde{\varepsilon} U^j - \left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right) g^{jm} \frac{\partial \tilde{\varepsilon}}{\partial \xi^m}\right] = c_{\varepsilon 1} f_1 G \frac{\tilde{\varepsilon}}{k} - c_{\varepsilon 2} f_2 \rho \frac{\tilde{\varepsilon}^2}{k} + \rho Y_c + P_{\varepsilon 3} \tag{C.62}$$

where the generation rate of turbulent kinetic energy, G, is given by:

$$G = -\rho g_{im} \overline{u^j u^m} \nabla_j U^i \tag{C.63}$$

the total dissipation rate at the wall, ε:

$$\varepsilon = \tilde{\varepsilon} + 2\nu g^{jm} \left(\frac{\partial k^{1/2}}{\partial \xi^m}\right) \left(\frac{\partial k^{1/2}}{\partial \xi^j}\right) \tag{C.64}$$

and the gradient production of dissipation rate, $P_{\varepsilon 3}$:

$$P_{\varepsilon 3} = 2\mu \nu_t g_{im} g_{jn} g^{lp} \left[\nabla_p \left(g^{no} \nabla_o U^m\right)\right] \left[\nabla_l \left(g^{jk} \nabla_k U^i\right)\right] \tag{C.65}$$

C.5.1 Physical Velocity Components

The notion of "physical" components was introduced in Section B.20. If one considers the simple case of a constant velocity field in plane polar coordinates $(r - \theta)$, one can show that the radial and circumferential velocity components, v_r and v_θ, are given by:

$$v_r = \frac{\partial r}{\partial t} = v^1 = \text{constant} \tag{C.66}$$

$$v_\theta = r \frac{\partial \theta}{\partial t} = r v^2 = \text{constant} \tag{C.67}$$

where v^1 and v^2 are the non-physical velocity components in the radial and circumferential directions, respectively. As one approaches the axis of the polar coordinate system $(r \to 0)$ the circumferential non-physical velocity component tends to infinity $\left(v^2 \to \infty\right)$. This can introduce serious errors in numerical calculations and, therefore, one must solve for the physical velocity components (in this case, v_r and v_θ) [85]. In general curvilinear coordinates, the physical contravariant velocity component, $U^{(i)}$, is obtained from:

$$U^i = \frac{U^{(i)}}{\sqrt{g_{ii}}} \quad \text{(no summation)} \tag{C.68}$$

C.5.2 RANS Equations Using Physical Velocity Vectors

Continuity

The continuity equation can be written with the velocity components in physical form, using Equations (C.56) and (B.133) :

$$\frac{\partial\rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial\xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(j)}\right) = 0 \tag{C.69}$$

which is expanded:

$$\frac{\partial\rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial\xi}\left(\frac{J}{\sqrt{g_{11}}}\rho U\right) + \frac{1}{J}\frac{\partial}{\partial\eta}\left(\frac{J}{\sqrt{g_{22}}}\rho V\right) + \frac{1}{J}\frac{\partial}{\partial\zeta}\left(\frac{J}{\sqrt{g_{33}}}\rho W\right) = 0 \tag{C.70}$$

where, as above, U, V and W denote the physical velocity components in the direction of the non-physical base vector components ξ, η and ζ respectively.

Scalar

Using Equations (C.50), (C.60) and (C.68), the scalar transport equation in nonorthogonal curvilinear coordinates can be written using physical velocity components as follows:

$$\boxed{\frac{\partial}{\partial t}(\rho\phi) + \frac{1}{J}\frac{\partial}{\partial\xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho\phi U^{(j)}\right) = \frac{1}{J}\frac{\partial}{\partial\xi^j}\left[J\left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi}\right)g^{jm}\frac{\partial\phi}{\partial\xi^m}\right] + S_\phi} \tag{C.71}$$

Momentum

An important difference with the scalar equation is that the divergence of a second-order tensor $(\rho\mathbf{U}\otimes\mathbf{U} - \mathbf{T})$ results in a vector, whereas in the scalar equation the divergence of $(\rho\mathbf{U}\phi - \mathbf{q})$ resulted in a scalar quantity. Therefore, the momentum equation can be written:

$$\left[\frac{\partial}{\partial t}\left(\rho U^i\right) + \nabla_j\left(\rho U^i U^j - \tau^{ij}\right) + g^{ij}\frac{\partial P}{\partial\xi^j}\right]\mathbf{g}_i = 0 \tag{C.72}$$

The base vector, $\mathbf{g}_i$, is written in "physical" form (i.e. as a unit vector) as follows:

$$\mathbf{g}_i = \sqrt{g_{ii}}\mathbf{g}_{(i)} \tag{C.73}$$

and the momentum equation becomes:

$$\left[\frac{\partial}{\partial t}\left(\rho U^i\right) + \nabla_j\left(\rho U^i U^j - \tau^{ij}\right) + g^{ij}\frac{\partial P}{\partial\xi^j}\right]\sqrt{g_{ii}}\mathbf{g}_{(i)} = 0 \tag{C.74}$$

It is important to keep the $\sqrt{g_{ii}}$ term in this expression so that the components are in terms of the physical base vectors. Without the $\sqrt{g_{ii}}$ term if one converts the equation into cylindrical polar coordinates, the axial, radial and *angular* (rather than tangential) momentum equations are obtained. Expanding the covariant derivative term $\left[\nabla_j\left(\rho U^i U^j - \tau^{ij}\right)\right]$ in the above expression using Equation (C.55) and writing the velocity components in physical form, $\left(U^i = U^{(i)}/\sqrt{g_{ii}}\right)$, the components of the momentum equation are given by:

$$\frac{\partial}{\partial t}\left(\rho U^{(i)}\right) + \underbrace{\sqrt{g_{ii}}\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{ii}g_{jj}}}\rho U^{(i)}U^{(j)}\right)}_{convection} + \rho U^{(j)}U^{(m)}\frac{\Gamma^i_{mj}\sqrt{g_{ii}}}{\sqrt{g_{jj}g_{mm}}}$$
$$= \left[-g^{ij}\frac{\partial P}{\partial \xi^j} + \underbrace{\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\tau^{ij}\right)}_{diffusion} + \tau^{mj}\Gamma^i_{mj}\right]\sqrt{g_{ii}} \qquad \text{(C.75)}$$

where there is no summation on the i index and it is assumed that the grid is not changing over time $\left(\partial\sqrt{g_{ii}}/\partial t = 0\right)$. The first underbraced term, denoted "convection", in the above expression can be written in a similar form to that given above for the ϕ-equation convection, using the quotient rule:

$$\sqrt{g_{ii}}\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{ii}g_{jj}}}\rho U^{(i)}U^{(j)}\right) = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(i)}U^{(j)}\right) - \frac{\rho U^{(i)}U^{(j)}}{\sqrt{g_{ii}g_{jj}}}\frac{\partial\sqrt{g_{ii}}}{\partial \xi^j} \qquad \text{(C.76)}$$

where, from Equation (B.121), the expression involving the derivative of the metric tensor can be written:

$$\frac{\partial\sqrt{g_{ii}}}{\partial \xi^j} = \frac{1}{2}\frac{1}{\sqrt{g_{ii}}}\frac{\partial g_{ii}}{\partial \xi^j} = \frac{1}{2}\frac{1}{\sqrt{g_{ii}}}\left(2\Gamma^m_{ij}g_{im}\right) = \frac{\Gamma^m_{ij}g_{im}}{\sqrt{g_{ii}}} \quad \text{(no summation on } i\text{)} \qquad \text{(C.77)}$$

The second underbraced term in Equation (C.75) can also be expanded, as follows:

$$\begin{aligned}\sqrt{g_{ii}}\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\tau^{ij}\right) &= \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\tau^{ij}\right) - \tau^{ij}\frac{\partial\sqrt{g_{ii}}}{\partial \xi^j} \\ &= \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\tau^{ij}\right) - \tau^{ij}\frac{\Gamma^m_{ij}g_{im}}{\sqrt{g_{ii}}}\end{aligned} \qquad \text{(C.78)}$$

Finally, substituting Equations (C.76), (C.77) and (C.78) into Equation (C.75):

$$\boxed{\frac{\partial}{\partial t}\left(\rho U^{(i)}\right) + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(i)}U^{(j)}\right) = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\tau^{ij}\right) + S^i_U} \qquad \text{(C.79)}$$

where:

$$\begin{aligned} S_U^i &= \rho U^{(i)} U^{(j)} \frac{\Gamma_{ij}^m g_{im}}{g_{ii}\sqrt{g_{jj}}} - \rho U^{(j)} U^{(m)} \frac{\Gamma_{mj}^i \sqrt{g_{ii}}}{\sqrt{g_{jj} g_{mm}}} - g^{ij}\sqrt{g_{ii}} \frac{\partial P}{\partial \xi^j} \\ &\quad - \tau^{ij} \frac{\Gamma_{ij}^m g_{im}}{\sqrt{g_{ii}}} + \tau^{mj} \Gamma_{mj}^i \sqrt{g_{ii}} \end{aligned} \tag{C.80}$$

There is no summation on index i in the above equations.

Stress Tensor, τ^{ij}

The non-physical stress tensor, τ^{ij}, for incompressible flow is given by:

$$\tau^{ij} = \mu \left(g^{jm} U^i_{,m} + g^{im} U^j_{,m} \right) - \rho \overline{u^i u^j} \tag{C.81}$$

The six independent stress components are expanded as follows:

$$\tau^{11} = 2\mu \left(g^{11} U_{,1} + g^{12} U_{,2} + g^{13} U_{,3} \right) - \rho \overline{uu} \tag{C.82}$$

$$\tau^{22} = 2\mu \left(g^{21} V_{,1} + g^{22} V_{,2} + g^{23} V_{,3} \right) - \rho \overline{vv} \tag{C.83}$$

$$\tau^{33} = 2\mu \left(g^{31} W_{,1} + g^{32} W_{,2} + g^{33} W_{,3} \right) - \rho \overline{ww} \tag{C.84}$$

$$\begin{aligned} \tau^{12} = \tau^{21} &= \mu \left(g^{21} U_{,1} + g^{22} U_{,2} + g^{23} U_{,3} \right. \\ &\quad \left. + g^{11} V_{,1} + g^{12} V_{,2} + g^{13} V_{,3} \right) - \rho \overline{uv} \end{aligned} \tag{C.85}$$

$$\begin{aligned} \tau^{13} = \tau^{31} &= \mu \left(g^{31} U_{,1} + g^{32} U_{,2} + g^{33} U_{,3} \right. \\ &\quad \left. + g^{11} W_{,1} + g^{12} W_{,2} + g^{13} W_{,3} \right) - \rho \overline{uw} \end{aligned} \tag{C.86}$$

$$\begin{aligned} \tau^{23} = \tau^{32} &= \mu \left(g^{31} V_{,1} + g^{32} V_{,2} + g^{33} V_{,3} \right. \\ &\quad \left. + g^{21} W_{,1} + g^{22} W_{,2} + g^{23} W_{,3} \right) - \rho \overline{vw} \end{aligned} \tag{C.87}$$

where strain rates $U_{,i}$, $V_{,i}$ and $W_{,i}$ are expanded in Section C.5.2. The non-physical components of the Reynolds stress $\left(-\rho \overline{u^i u^j} \right)$ are obtained from Equation (C.59) for an incompressible flow, as follows:

$$-\rho \overline{u^i u^j} = \mu_t \left(g^{jm} U^i_{,m} + g^{im} U^j_{,m} \right) - \frac{2}{3} g^{ij} \rho k \tag{C.88}$$

which is expanded:

$$-\rho\overline{uu} = 2\mu_t \left(g^{11}U_{,1} + g^{12}U_{,2} + g^{13}U_{,3}\right) - \frac{2}{3}g^{11}\rho k \tag{C.89}$$

$$-\rho\overline{vv} = 2\mu_t \left(g^{21}V_{,1} + g^{22}V_{,2} + g^{23}V_{,3}\right) - \frac{2}{3}g^{22}\rho k \tag{C.90}$$

$$-\rho\overline{ww} = 2\mu_t \left(g^{31}W_{,1} + g^{32}W_{,2} + g^{33}W_{,3}\right) - \frac{2}{3}g^{33}\rho k \tag{C.91}$$

$$\begin{aligned} -\rho\overline{uv} = -\rho\overline{vu} &= \mu_t \left(g^{21}U_{,1} + g^{22}U_{,2} + g^{23}U_{,3}\right. \\ &\quad \left.+g^{11}V_{,1} + g^{12}V_{,2} + g^{13}V_{,3}\right) - \frac{2}{3}g^{12}\rho k \end{aligned} \tag{C.92}$$

$$\begin{aligned} -\rho\overline{uw} = -\rho\overline{wu} &= \mu_t \left(g^{31}U_{,1} + g^{32}U_{,2} + g^{33}U_{,3}\right. \\ &\quad \left.+g^{11}W_{,1} + g^{12}W_{,2} + g^{13}W_{,3}\right) - \frac{2}{3}g^{13}\rho k \end{aligned} \tag{C.93}$$

$$\begin{aligned} -\rho\overline{vw} = -\rho\overline{wv} &= \mu_t \left(g^{31}V_{,1} + g^{32}V_{,2} + g^{33}V_{,3}\right. \\ &\quad \left.+g^{21}W_{,1} + g^{22}W_{,2} + g^{23}W_{,3}\right) - \frac{2}{3}g^{23}\rho k \end{aligned} \tag{C.94}$$

Velocity Gradient, $U^i_{,j}$

It was shown above that the strain-rate in Cartesian coordinates $\left(\partial U^i/\partial x^j\right)$ transforms into the following expression in non-orthogonal curvilinear coordinates:

$$\frac{\partial U^i}{\partial x^j} \rightarrow \nabla_j U^i \equiv U^i_{,j} = \frac{\partial U^i}{\partial \xi^j} + U^m \Gamma^i_{mj} \tag{C.95}$$

which is expressed in terms of the physical velocity component $U^{(i)}$:

$$U^i_{,j} = \frac{\partial}{\partial \xi^j}\left(\frac{U^{(i)}}{\sqrt{g_{ii}}}\right) + \frac{U^{(m)}}{\sqrt{g_{mm}}}\Gamma^i_{mj} \tag{C.96}$$

The strain-rate is expanded in 3-D curvilinear coordinates, as follows:

$$U_{,1} = \frac{\partial}{\partial \xi}\left(\frac{U}{\sqrt{g_{11}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^1_{11} + \frac{V}{\sqrt{g_{22}}}\Gamma^1_{21} + \frac{W}{\sqrt{g_{33}}}\Gamma^1_{31} \tag{C.97}$$

$$U_{,2} = \frac{\partial}{\partial \eta}\left(\frac{U}{\sqrt{g_{11}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^1_{12} + \frac{V}{\sqrt{g_{22}}}\Gamma^1_{22} + \frac{W}{\sqrt{g_{33}}}\Gamma^1_{32} \tag{C.98}$$

$$U_{,3} = \frac{\partial}{\partial \zeta}\left(\frac{U}{\sqrt{g_{11}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^1_{13} + \frac{V}{\sqrt{g_{22}}}\Gamma^1_{23} + \frac{W}{\sqrt{g_{33}}}\Gamma^1_{33} \tag{C.99}$$

$$V_{,1} = \frac{\partial}{\partial \xi}\left(\frac{V}{\sqrt{g_{22}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^2_{11} + \frac{V}{\sqrt{g_{22}}}\Gamma^2_{21} + \frac{W}{\sqrt{g_{33}}}\Gamma^2_{31} \tag{C.100}$$

$$V_{,2} = \frac{\partial}{\partial \eta}\left(\frac{V}{\sqrt{g_{22}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^2_{12} + \frac{V}{\sqrt{g_{22}}}\Gamma^2_{22} + \frac{W}{\sqrt{g_{33}}}\Gamma^2_{32} \tag{C.101}$$

$$V_{,3} = \frac{\partial}{\partial \zeta}\left(\frac{V}{\sqrt{g_{22}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^2_{13} + \frac{V}{\sqrt{g_{22}}}\Gamma^2_{23} + \frac{W}{\sqrt{g_{33}}}\Gamma^2_{33} \tag{C.102}$$

$$W_{,1} = \frac{\partial}{\partial \xi}\left(\frac{W}{\sqrt{g_{33}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^3_{11} + \frac{V}{\sqrt{g_{22}}}\Gamma^3_{21} + \frac{W}{\sqrt{g_{33}}}\Gamma^3_{31} \tag{C.103}$$

$$W_{,(2)} = \frac{\partial}{\partial \eta}\left(\frac{W}{\sqrt{g_{33}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^3_{12} + \frac{V}{\sqrt{g_{22}}}\Gamma^3_{22} + \frac{W}{\sqrt{g_{33}}}\Gamma^3_{32} \tag{C.104}$$

$$W_{,(3)} = \frac{\partial}{\partial \zeta}\left(\frac{W}{\sqrt{g_{33}}}\right) + \frac{U}{\sqrt{g_{11}}}\Gamma^3_{13} + \frac{V}{\sqrt{g_{22}}}\Gamma^3_{23} + \frac{W}{\sqrt{g_{33}}}\Gamma^3_{33} \tag{C.105}$$

where $U^{(i)} = (U, V, W)$ denote the physical velocity components and $\xi^i = (\xi, \eta, \zeta)$ the non-physical contravariant components of the covariant base vectors $\mathbf{g}_i = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$.

C.5.3 Examination of Curvilinear Transport Equations

The momentum equation in curvilinear coordinates is given by:

$$\frac{\partial}{\partial t}\left(\rho U^{(i)}\right) + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(i)}U^{(j)}\right) = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\tau^{ij}\right) + S^i_U \tag{C.106}$$

where the source term, S^i_U, is given by Equation (C.80). The diffusion term in the above equation can be rearranged so that it follows the same format as the diffusion term in the scalar equation (Equation C.71). The stress tensor for an incompressible flow, using a linear eddy-viscosity model, can be written:

$$\begin{aligned} \tau^{ij} &= \mu_{eff}\left(g^{jm}\nabla_m U^i + g^{im}\nabla_m U^j\right) - \frac{2}{3}g^{ij}\rho k \\ &= \mu_{eff}\left[g^{jm}\left(\frac{\partial U^i}{\partial \xi^m} + \Gamma^i_{km}U^k\right) + g^{im}\left(\frac{\partial U^j}{\partial \xi^m} + \Gamma^j_{km}U^k\right)\right] - \frac{2}{3}g^{ij}\rho k \end{aligned} \tag{C.107}$$

Substituting the above equation into Equation (C.106), the first component of the diffusion term is given by:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\mu_{eff}g^{jm}\frac{\partial U^i}{\partial \xi^m}\right) \tag{C.108}$$

Writing the velocity component U^i in terms of the physical component $U^{(i)}$:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left[\sqrt{g_{ii}}J\mu_{eff}g^{jm}\frac{\partial}{\partial \xi^m}\left(\frac{U^{(i)}}{\sqrt{g_{ii}}}\right)\right] \tag{C.109}$$

and expanding the inner-derivative using the quotient rule, one obtains:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\mu_{eff}g^{jm}\frac{\partial U^{(i)}}{\partial \xi^m} - \frac{J}{\sqrt{g_{ii}}}\mu_{eff}g^{jm}U^{(i)}\frac{\partial\sqrt{g_{ii}}}{\partial \xi^m}\right) \tag{C.110}$$

The first term in the above Equation is of a similar format to the diffusion term in the scalar equation diffusion term, which was given as (Equation C.71):

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left[J\left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi}\right)g^{jm}\frac{\partial \phi}{\partial \xi^m}\right] \tag{C.111}$$

Checking the Formulation of Convection

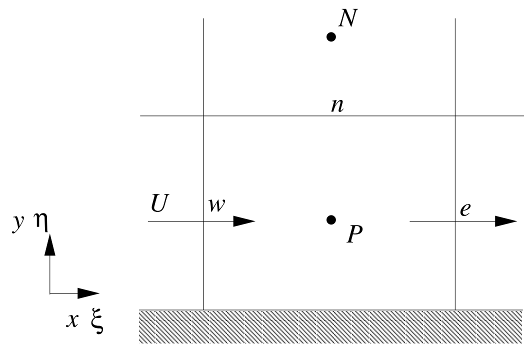

Figure C.1: Cartesian grid cell with mean velocity U parallel to the x-axis

In the preceding section it was shown that the transport equations for momentum and scalar can be expressed in the form:

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho\phi U^{(j)}\right) = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\Gamma g^{jm}\frac{\partial \phi}{\partial \xi^m}\right) + S_\phi \tag{C.112}$$

where ϕ represents a scalar parameter or the physical velocity component, Γ denotes the relevant diffusivity and S_ϕ includes the pressure gradient and curvature source terms in the momentum equation. To confirm that the convection term in the above equation has been derived correctly, one can show that by integrating the convection term over a control volume one obtains an expression involving mass fluxes of parameter ϕ through the cell faces. The convection of scalar ϕ in the ξ-direction is discretized and integrated over a 2-D physical cell volume (see Figure C.1) as follows:

$$\begin{aligned}
&\int \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho\phi U^{(j)}\right) \mathrm{d}Vol \\
&= \frac{1}{\Delta\xi_{ew}}\left[\left(\frac{J}{\sqrt{g_{11}}}\rho\phi U\right)_e - \left(\frac{J}{\sqrt{g_{11}}}\rho\phi U\right)_w\right]\Delta\xi_{ew}\Delta\eta_{ns}
\end{aligned} \tag{C.113}$$

where U is the physical velocity component in the ξ-direction and $\Delta Vol = J\Delta\xi\Delta\eta$ is the cell volume (equivalent to an area in 2-D). Each of the dimensionless distances in computational space has unit magnitude:

$$\Delta\xi_{ew} = \Delta\eta_{ns} = 1 \tag{C.114}$$

the Jacobian is therefore the cell volume:

$$J = \Delta Vol \tag{C.115}$$

The metric tensor $(g_{11})_e$ is equivalent to square of the physical distance between the nodes P and E:

$$\begin{aligned}
(g_{11})_e &= \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 \\
&\approx \left(\frac{\Delta x_{EP}}{\Delta \xi_{EP}}\right)^2 + \left(\frac{\Delta y_{EP}}{\Delta \xi_{EP}}\right)^2 \\
&\approx (\Delta x_{EP})^2
\end{aligned} \tag{C.116}$$

and the convective flux through the eastern cell faces is therefore given by:

$$\left(\frac{J}{\sqrt{g_{11}}}\rho\phi U\right)_e = \left(\frac{\Delta Vol}{\Delta x_{EP}}\rho\phi U\right)_e = (\Delta y_{ns}\rho\phi U)_e \tag{C.117}$$

where the velocity U_e is normal to the eastern cell face which has area Δy_{ns}. Therefore, $(\Delta y_{ns}\rho\phi U)_e$ represents the mass flux of ϕ through the eastern cell face.

Checking the Formulation of Diffusion

The second check on the derivation of the curvilinear transport equations involves the diffusion term. Figure C.2 shows two grid arrangements with equal spacing of grid-lines in both the vertical y-axis and the horizontal x-axis. In the first grid, the base vectors are aligned to a Cartesian grid (in which case $\xi = x$ and $\eta = y$) but in the second grid the η-axis is skewed from the vertical. A gradient in ϕ in the vertical y-direction is imposed and it should be demonstrated that equal diffusion of ϕ is obtained using the two different grids (assuming

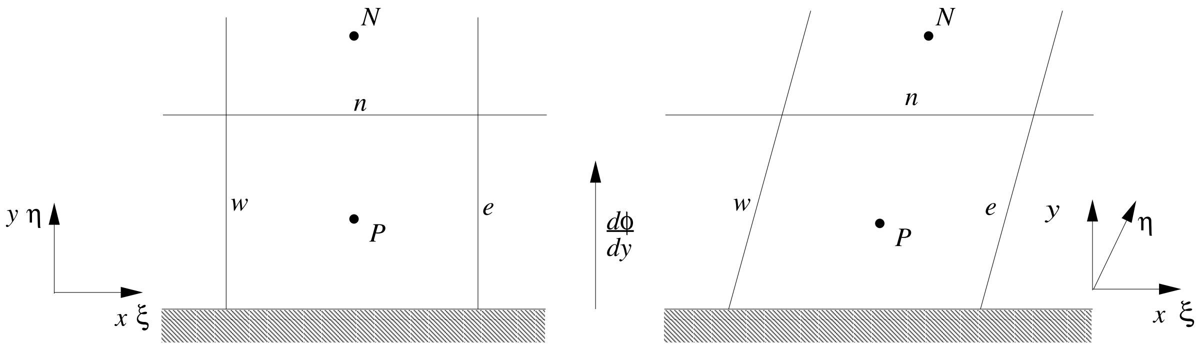

Figure C.2: Cartesian and skew grid arrangements with vertical gradient of ϕ

there to be no gradient of ϕ in the x-direction). The diffusion term is given by:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J g^{jm}\Gamma\frac{\partial \phi}{\partial \xi^m}\right) \tag{C.118}$$

In the 2-D case considered here, the diffusion of ϕ in the vertical direction is calculated from:

$$\frac{1}{J}\frac{\partial}{\partial \eta}\left(J g^{22}\Gamma\frac{\partial \phi}{\partial \eta}\right) \tag{C.119}$$

As before, the Jacobian in 2-D is equivalent to the cell area:

$$J = Area = \Delta x_{ew}\Delta y_{ns} = \Delta x_{ew}\Delta y_{NP} \tag{C.120}$$

The contravariant metric tensor, g^{22}, is obtained from matrix inversion:

$$\begin{aligned} g^{ij} = \frac{1}{g}G_{ij} &= \frac{1}{J^2}\left[cof\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}\right]^T \\ &= \frac{1}{J^2}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{bmatrix} \end{aligned} \tag{C.121}$$

$$g^{22} = \frac{1}{J^2}g_{11} = \frac{1}{J^2}\left(\Delta x_{ew}\right)^2 \tag{C.122}$$

Substituting Equations (C.120) and (C.122) into (C.119):

$$\frac{1}{J}\frac{\partial}{\partial \eta}\left(J g^{22}\Gamma\frac{\partial \phi}{\partial \eta}\right) = \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\left(\Delta x_{ew}\right)^2}{J}\Gamma\frac{\partial \phi}{\partial \eta}\right) \tag{C.123}$$

$$= \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\Delta x_{ew}}{\Delta y_{NP}}\Gamma\frac{\partial \phi}{\partial \eta}\right) \tag{C.124}$$

The diffusive fluxes obtained by discretizing and integrating the above equation are of the form:

$$\frac{\Gamma A}{L} \tag{C.125}$$

where where A is the cell-face area (equivalent in 2-D to Δx_{ew}) and L is the vertical distance between adjacent nodes (i.e. Δy_{TP}). The diffusion of ϕ in the vertical direction is therefore identical whether a Cartesian or skew grid is employed.

C.5.4 Non-Conservative Convection

Scalar

Convection of scalar ϕ in conservative form was shown earlier to be given by:

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho\phi U^{(j)}\right) \tag{C.126}$$

Expanding this using the product rule:

$$\phi\frac{\partial \rho}{\partial t} + \rho\frac{\partial \phi}{\partial t} + \frac{\rho U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial \phi}{\partial \xi^j} + \frac{\phi}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(j)}\right) \tag{C.127}$$

Continuity is expressed in curvilinear coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\Delta}{\Delta \xi^j}\left(\rho U^j\right) = 0 \tag{C.128}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(j)}\right) = 0 \tag{C.129}$$

Therefore, Equation (C.127) simplifies to:

$$\boxed{\rho\frac{\partial \phi}{\partial t} + \frac{\rho U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial \phi}{\partial \xi^j}} \tag{C.130}$$

Momentum

Convection of momentum can also be obtained in non-conservative form by expanding the conservative form and canceling terms using continuity. The conservative convection term was shown earlier to be given by:

$$\frac{\partial}{\partial t}\left(\rho U^{(i)}\right) + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{J}{\sqrt{g_{jj}}}\rho U^{(i)} U^{(j)}\right) \tag{C.131}$$

This can be rearranged to give the non-conservative form:

$$\boxed{\rho\frac{\partial U^{(i)}}{\partial t} + \frac{\rho U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial U^{(i)}}{\partial \xi^j}} \tag{C.132}$$

C.5.5 Alternative Approach to Derivation

In the preceding analysis, it has been shown that the convection term in the momentum equations can be written in curvilinear coordinates, as follows:

$$\begin{aligned}(\mathbf{U}\cdot\nabla)\,\mathbf{U} &= \nabla_j\left(U^iU^j\right)\mathbf{g}_{(i)} \\ &= \left[\frac{U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial U^{(i)}}{\partial \xi^j} - U^{(i)}U^{(j)}\frac{\Gamma^m_{ij}g_{im}}{g_{ii}\sqrt{g_{jj}}} + U^{(j)}U^{(m)}\frac{\Gamma^i_{mj}\sqrt{g_{ii}}}{\sqrt{g_{jj}g_{mm}}}\right]\mathbf{g}_{(i)}\end{aligned} \tag{C.133}$$

The above expression was obtained by converting the Cartesian convection term into curvilinear coordinates. To check that the derivation is correct, one can re-derive the convection term by starting directly from the vector form, $(\mathbf{U}\cdot\nabla)\,\mathbf{U}$, as follows:

$$\begin{aligned}(\mathbf{U}\cdot\nabla)\,\mathbf{U} &= U^j\mathbf{g}_j\cdot\mathbf{g}^k\frac{\partial}{\partial\xi^k}\left(U^i\mathbf{g}_i\right) \\ &= U^j\delta^k_j\frac{\partial}{\partial\xi^k}\left(U^i\mathbf{g}_i\right) \\ &= U^j\frac{\partial}{\partial\xi^j}\left(U^i\mathbf{g}_i\right)\end{aligned} \tag{C.134}$$

The covariant derivative of a vector is given by:

$$\frac{\partial}{\partial\xi^j}\left(U^i\mathbf{g}_i\right) = \left(\frac{\partial U^i}{\partial\xi^j} + U^m\Gamma^i_{mj}\right)\mathbf{g}_i \tag{C.135}$$

and therefore we obtain:

$$(\mathbf{U}\cdot\nabla)\,\mathbf{U} = \left(U^j\frac{\partial U^i}{\partial\xi^j} + U^jU^m\Gamma^i_{mj}\right)\mathbf{g}_i \tag{C.136}$$

Using physical velocity components $U^{(i)}$, where $U^i = U^{(i)}/\sqrt{g_{ii}}$, and physical base vectors $\mathbf{g}_{(i)}$, where $\mathbf{g}_i = \sqrt{g_{ii}}\mathbf{g}_{(i)}$:

$$(\mathbf{U}\cdot\nabla)\,\mathbf{U} = \left[\underbrace{\sqrt{g_{ii}}\frac{U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial}{\partial\xi^j}\left(\frac{U^{(i)}}{\sqrt{g_{ii}}}\right)} + U^{(j)}U^{(m)}\frac{\Gamma^i_{mj}\sqrt{g_{ii}}}{\sqrt{g_{jj}g_{mm}}}\right]\mathbf{g}_{(i)} \tag{C.137}$$

and using the quotient rule to rearrange the underbraced term:

$$\sqrt{g_{ii}}\frac{U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial}{\partial \xi^j}\left(\frac{U^{(i)}}{\sqrt{g_{ii}}}\right) = \frac{U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial U^{(i)}}{\partial \xi^j} - \frac{U^{(i)}U^{(j)}}{\sqrt{g_{ii}g_{jj}}}\frac{\partial\sqrt{g_{ii}}}{\partial \xi^j} \tag{C.138}$$

From Equation (C.77), the gradient term, $\partial\sqrt{g_{ii}}/\partial\xi^j$, can be expressed in terms of the Christoffel symbols and the convection term becomes:

$$(\mathbf{U}\cdot\nabla)\,\mathbf{U} = \left[\frac{U^{(j)}}{\sqrt{g_{jj}}}\frac{\partial U^{(i)}}{\partial \xi^j} - U^{(i)}U^{(j)}\frac{\Gamma^m_{ij}g_{im}}{g_{ii}\sqrt{g_{jj}}} + U^{(j)}U^{(m)}\frac{\Gamma^i_{mj}\sqrt{g_{ii}}}{\sqrt{g_{jj}g_{mm}}}\right]\mathbf{g}_{(i)} \tag{C.139}$$

This is identical to the expression which was derived earlier (Equation C.133).